[MATHS(AH) - EX 2010-2011

NATIONAL
QUALIFICATIONS
2010 – 2011

TIME: 3 HOURS

MATHEMATICS
ADVANCED HIGHER
Units 1 and 2

Estimate Examination Paper

Read carefully

1. Calculators may be used in this paper.
2. Candidate should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**

Answer all the questions.

Marks

Answer all the questions

1. (*a*) Given $f(x) = \ln(x+1) \cdot 5x^2$, where $x > -1$, obtain $f'(x)$. **3**

(*b*) For $y = \dfrac{5 - x^2}{1 + x^3}$, determine $\dfrac{dy}{dx}$ in simplified form. **3**

2. Define $S_n(a)$ by $S_n(a) = \dfrac{\pi}{2} + \dfrac{3a\pi}{4} + a^2\pi + ... + \dfrac{(n+1)\pi a^{n-1}}{4}$.

Calculate $S_{16}(1)$. **3**

3. Use Gaussian elimination to determine the values of *k* that give a unique solution for the equations

$$\begin{aligned} x - y + 2z &= 5 \\ x + 2y - z &= -6 \\ 2x - 3y + kz &= 0 \end{aligned}$$

5

What value of *k* would give rise to an inconsistency? **1**

4. Write down and simplify the general term in the expansion of $\left(2x^3 + \dfrac{3}{x^2}\right)^5$. **3**

Hence, or otherwise, obtain the term independent of *x*. **2**

5. Use the substitution $u = \ln x$ to evaluate $\displaystyle\int_1^{\sqrt{e}} \frac{1}{\sqrt{x^2 - x^2(\ln x)^2}}\, dx$, $x > 0$. **6**

Marks

6. A curve is defined by the equations

$$x = 3\sin t, \qquad y = -2\cos t, \qquad (0 \le t \le 2\pi).$$

Use parametric differentiation to find $\frac{dy}{dx}$ in terms of t. **2**

Find the equation of the tangent to the curve at the point where $t = \frac{\pi}{3}$. **3**

7. Determine whether the function $f(x) = 2x^5 \tan 3x$, $-\frac{\pi}{3} < x < \frac{\pi}{3}$ is odd, even or neither. **3**

8. A geometric sequence is defined by $u_n = ar^{n-1}$, where $a \in R$ and $0 < r < 1$.
The sum of the first 3 terms in this geometric series is $\frac{95}{9}$ and the sum to infinity of this geometric series is 15.
Find the values of *a* and *r*. **5**

9. A number, n, is defined as $n(x) = x^3 - x$ where x is a positive integer and $x \ge 2$.
Prove that *n* is always divisible by 3. **6**

10. The curve $y = x^{5x^3-2}$ is defined for $x > 0$. Obtain the values of *y* and $\frac{dy}{dx}$ at the point where $x = 1$. **5**

11. Show that *i* is a solution of $2z^3 - 3z^2 + 2z = 3$.
Hence find all solutions. **4**

[Turn over

Marks

12. The Malthus Equation is a basic model used to predict population increases and decreases.

It states $\frac{dP}{dt} = kP$ where k is a positive constant,

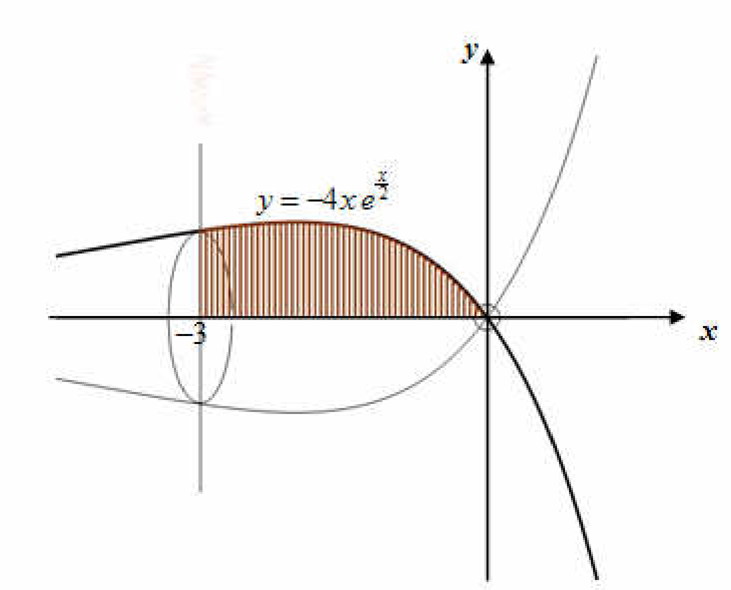
P is the population and t is the time in years.

The initial population of a town was 5 000 people.

Five years later, the town's population had grown by 1 200 people.

Find the value of k. **6**

13. The design for a solid Christmas decoration is made by rotating the area bound by the curve $y = -4xe^{\frac{x}{2}}$ and the x-axis about the x-axis by 2π radians between the origin and $x = -3$ as shown on the diagram below.



Find the volume of plastic required to make the Christmas decoration. **9**

14. (*a*) Plot the complex number $1+\sqrt{3}\,i$ on an Argand diagram. **2**

(*b*) $z^3 = 1+\sqrt{3}\,i$. Find all the roots of z^3, expressing each in the form $z = r(\cos\theta + i\sin\theta)$, clearly stating the values of r and θ.

Plot z on the Argand diagram used in part (a). **7**

Marks

15. (*a*) Prove by induction that $\sum_{r=1}^{n} 4r(r-2) = \frac{2n}{3}(n+1)(2n-5)$

for all natural numbers $n \geq 1$. **4**

(*b*) Hence evaluate $\sum_{r=10}^{25} 4r(r-2)$. **2**

16. Given the equation $3x^2 + 6x - 6xy - 2y + 3y^2 = 5$ of a curve, obtain the *x*-coordinate for the point at which the curve has a horizontal tangent. **7**

17. Express $\frac{5x}{(x-2)^2}$ in partial fractions. **3**

A curve is defined by $y = \frac{5x}{(x-2)^2}$, $(x \neq 2)$.

(i) Write down equations for its asymptotes. **2**

(ii) Find the stationary point and justify its nature. **4**

(iii) Sketch the curve showing clearly the features found in (i) and (ii). **2**

Total: 102 marks

[*END OF QUESTION PAPER*]

ADDITIONAL QUESTIONS FOR UNIT 3

Notes for Inserting Unit 3 questions:

1. As the main paper has more Unit 2 questions it may be beneficial to replace some of these with the unit 3 questions.
   Unit 2 questions are as follows: Questions 2, 8, 9, 10, 11, 12, 14, 15 and 16
2. If adding in unit 3, may we suggest A1, D1 and E1 be included for appeals purposes. We try to replicate the actual exam as much as possible so these longer question would be good for this.

A. VECTORS

Marks

A1. (*a*) Find the equation of the plane Π which contains the points P(2, 1, 0), Q(–1, 0, 0) and R(1, 1, –1). **4**

(*b*) Calculate the point of intersection of the plane Π and the line L,

$$\frac{x+2}{3} = y-1 = \frac{z+8}{2}.$$

Find the size of the acute angle between the line L and the plane Π. **6**

B. MATRIX ALGEBRA

B1. [*This question should only be used if question 3 from the units 1 & 2 section is removed*]

A matrix is defined as $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ -4 & 1 & 1 \end{pmatrix}$.

Show that matrix A has an inverse, A^{-1}, and use elementary row operations to find the inverse matrix. **6**

B2. The matrix P is such that $P^2 = 3P - 5I$ where I is the corresponding identity matrix. Find integers a and b such that

$$P^4 = aP + bI .$$

4

B3. Write down the 2×2 matrix A representing a rotation of $\frac{\pi}{3}$ radians about the origin in an anticlockwise direction and the 2×2 matrix B representing a reflection in the y-axis.

Hence, show that the image of the point (x, y) under the transformation A followed by the transformation B is $\left(-\frac{x-py}{2}, \frac{px+y}{2}\right)$, stating the value of p. **5**

C. SEQUENCES & SERIES: MACLAURIN SERIES, ITERATION and CONVERGENCE

C1. Find the Maclaurin expansion of

$$f(x) = \ln(1+\sin x),\ 0 < x \le \frac{\pi}{2}$$

as far as the term in x^3. **5**

C2. The equation $y = x^3 + x - 5$ has root between $x = 1$ and $x = 2$.

Using the iterative recurrence formula $x_{n+1} = \sqrt[3]{5-x}$ and $x_0 = 1$ find the value of the root correct to 3 decimal places. **5**

D. DIFFERENTIAL EQUATIONS

D1. Solve the differential equation

$$2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 3y = 2\sin x\ .$$

given that there is a stationary point at (0, 1). **10**

E. NUMBER THEORY & PROOF: EUCLIDEAN ALGORITHM

E1. [*This would be as an 'add on' to question 15 from the Units1 & 2 section*]

(c) Use direct proof to show that $\sum_{r=1}^{n} 4r(r-2) = \frac{2n}{3}(n+1)(2n-5)$. **4**

E2. Use the Euclidean Algorithm to find integers *x* and *y* such that **4**

$$159x + 127y = 1.$$

[*END OF ADDITIONAL QUESTIONS*]

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[MATH(AH)MS - 2011]

NATIONAL
QUALIFICATIONS
2011

Marking Instructions

ADVANCED HIGHER MATHEMATICS

The security of this examination paper requires that it is withdrawn from candidates after the examination and also after any discussion of the candidates' results. This will ensure that the paper continues to be secure for your centre and others during presentation year 2010/2011. Any appeals made based on this paper will assume that these security precautions are in place.

Advanced Higher Mathematics
Marking Instructions

General Marking Principles
These principles describe the approach taken when marking Advanced Higher Mathematics papers.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:
- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

Distribution of marks
Candidates will be expected to answer all of the questions. There will be a total of 102 marks for the paper.

The below suggested marking thresholds are based on an unaltered paper for units 1 and 2.

If inserting unit 3 questions then the below marking thresholds may only be used if:

1] the total number of A marks and the total number of B marks is the same or greater
2] each of the three units has at least 30% of the marks

If either or both of the above criteria are not met, the cutoffs should be adjusted *upwards*

<u>Suggested Marking Thresholds</u>

Mark	Grade
90%	A1
75%	A
63%	B
50%	C
45%	D

No		Analysis			Question	Illustrations of evidence for awarding each mark	Marks
		Unit / Outcome	Marks at levels				
			A/B	C			
1.	*(a)*	1.2		3	Given $f(x) = \ln(x+1) \cdot 5x^2$, where $x > -1$, obtain $f'(x)$.	• Using the product rule $f'(x) = \frac{d}{dx}\left(\ln(x+1)\right) \cdot 5x^2 + \ln(x+1) \cdot \frac{d}{dx}\left(5x^2\right)$ • Correct first term $\frac{5x^2}{x+1} + ...$ • Correct second term $..... + \ln(x+1) \cdot 10x$	**3**
	(b)	1.2		3	For $y = \frac{5-x^2}{1+x^3}$, determine $\frac{dy}{dx}$ in simplified form.	• Using the quotient rule $\frac{\frac{d}{dx}\left(5-x^2\right)\left(1+x^3\right) - \left(5-x^2\right)\frac{d}{dx}\left(1+x^3\right)}{\left(1+x^3\right)^2}$ • Correct derivatives of terms $\frac{-2x\left(1+x^3\right) - \left(5-x^2\right)3x^2}{\left(1+x^3\right)^2}$ • Answer simplified and stated $\frac{x^4 - 15x^2 - 2x}{\left(1+x^3\right)^2}$	**3**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
2.	2.4		3	Define $S_n(a)$ by $S_n(a) = \frac{\pi}{2} + \frac{3a\pi}{4} + a^2\pi + ... + \frac{(n+1)\pi a^{n-1}}{4}$. Calculate $S_{16}(1)$.	• Obtain correct series and identify common difference and initial term $S_n(1) = \frac{\pi}{2} + \frac{3\pi}{4} + \pi + ... + \frac{(n+1)\pi}{4}$ and so $d = \frac{\pi}{4}$, $u_1 = a = \frac{\pi}{2}$ • Know correct formula for finding the sum of an arithmetic series $S_{16}(1) = \frac{16}{2}\left(2 \times \frac{\pi}{2} + (16-1) \times \frac{\pi}{4}\right)$ • The sum to 16 terms $8\left(\pi + \frac{15\pi}{4}\right) = 38\pi$	**3**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Ma6rks
	Unit / Outcome	Marks at levels				
		A/B	C			
3.	1.5		5 1	Use Gaussian elimination to determine the values of k that give a unique solution for the equations $x - y + 2z = 5$ $x + 2y - z = -6$ $2x - 3y + kz = 0$ What value of k would give rise to an inconsistency?	• For a structured approach: matrix form $\begin{pmatrix} 1 & -1 & 2 & 5 \\ 1 & 2 & -1 & -6 \\ 2 & -3 & k & 0 \end{pmatrix}$ • For simplifying matrix $\begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & -1 & k-4 & -10 \end{pmatrix}$ • For upper triangular form $\begin{pmatrix} 1 & -1 & 2 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & 3k-15 & -43 \end{pmatrix}$ • Know the necessary criteria $3k - 15 \neq 0$ • Correctly calculate answer $k \neq 5$ • Know criteria for an inconsistency $k = 5$	**5** **1**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
4.	1.1	3		Write down and simplify the general term in the expansion of $\left(2x^3+\dfrac{3}{x^2}\right)^5$.	• Correct form/use of notation $\dbinom{5}{r}\left(2x^3\right)^{\boxed{r}}\left(\dfrac{3}{x^2}\right)^{\boxed{5-r}}$ • Correct powers $\dbinom{5}{r}\left(2x^3\right)^{r}\left(\dfrac{3}{x^2}\right)^{5-r}$ • Simplifying $\dbinom{5}{r}2^r\cdot 3^{5-r}\cdot x^{5r-10}$	**3**
	1.1	2		Hence, or otherwise, obtain the term independent of x.	• Identify correct term $5r-10=0 \quad \therefore\ r=2$ • Evaluate term correctly $\dbinom{5}{2}\cdot 2^2\cdot 3^{5-2}=1080$	**2**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
5.	1.3	4	**2**	Use the substitution $u = \ln x$ to evaluate $$\int_1^{\sqrt{e}} \frac{1}{\sqrt{x^2 - x^2(\ln x)^2}}\, dx,\ x > 0.$$	• Differentiate u and get expression for dx $\frac{du}{dx} = \frac{1}{x} \quad \therefore dx = x\ du$ • Simplify expression $\int_1^{\sqrt{e}} \frac{1}{x\sqrt{1-(\ln x)^2}}\, dx$ • Make all relevant substitutions $\int_0^{1/2} \frac{1}{x\sqrt{1-(u)^2}}\, x\ du$ • Simplify $\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\, du$ • Integrate $\left[\sin^{-1} u\right]_0^{1/2}$ • Evaluate $\frac{\pi}{3}$	**6**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
6.	1.1	2		A curve is defined by the equations $x = 3\sin t$ $y = -2\cos t$, $(0 \le t \le 2\pi)$. Use parametric differentiation to find $\frac{dy}{dx}$ in terms of t.	• Differentiate x and y $\frac{dx}{dt} = 3\cos t$ $\frac{dy}{dt} = 2\sin t$ • Obtain correct expression for $\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{2}{3}\tan t$	**2**
			3	Find the equation of the tangent to the curve at the point where $t = \frac{\pi}{3}$.	• Calculate gradient $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$ • Find x and y values $x = \frac{3\sqrt{3}}{3}$ $y = -1$ • Equation of tangent $y + 1 = \frac{2\sqrt{3}}{3}\left(x - \frac{3\sqrt{3}}{2}\right)$	**3**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
7.	1.4		3	Determine whether the function $f(x) = 2x^5 \tan 3x$, $-\frac{\pi}{3} < x < \frac{\pi}{3}$ is odd, even or neither.	• Know how test for odd/even $f(-x) = 2(-x)^5 \cdot \tan(-3x)$ • Compare results $\ldots = 2x^5 \tan 3x = f(x)$ • Correct conclusion $f(x)$ is even	**3**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
8.	2.4		5	A geometric sequence is defined by $u_n = ar^{n-1}$, where $a \in R$ and $0 < r < 1$. The sum of the first 3 terms in this geometric series is $\frac{95}{9}$ and the sum to infinity of this geometric series is 15. Find the values of *a* and *r*.	• Expression for sum to 3 terms $\frac{a(1-r^3)}{1-r} = \frac{95}{9}$ • Expression for sum to infinity terms $\frac{a}{1-r} = 15$ • Create equation $\frac{9a(1-r^3)}{95(1-r)} = \frac{a}{15(1-r)}$ • Simplify $135(1-r^3) = 95$ • Solve to get values of *r* then *a*. $r = \frac{2}{3},\ a = 5$	**5**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
9.	2.5		3	A number, n, is defined as $n(x) = x^3 - x$ where x is a positive integer and $x \geq 2$. Prove that *n* is always divisible by 3.	• Strategy: valid/exhaustive method of proof Induction: $x = 2 \Rightarrow n(x) = 6$ so true for $x = 2$ • Process Assume true for $x = k \therefore k^3 - k = 3a$ where $a \in N$ • Continue proof For $x = k + 1$ $(k+1)^3 - (k+1) = (k+1)\left((k+1)^2 - 1\right)$ • Continue manipulation $\ldots = (k+1)(k^2 + 2k)$ $= k(k+1)(k+2)$ • Complete manipulation $\ldots = (k^3 + k)(k+2)$ $= 3a(k+2)$ • and so, if true for $x = k$ then also true for $x = k + 1$ and since true for $x = 2$, $x^3 - x$ is divisible by 3 $\forall x \geq 2$.	**6**

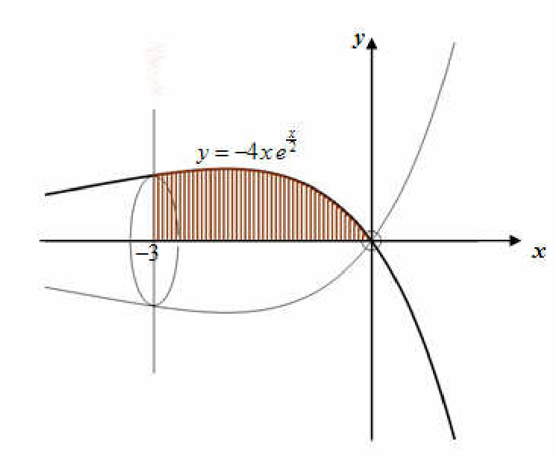
[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
10.	2.1	5		The curve $y = x^{5x^3-2}$ is defined for $x > 0$. Obtain the values of y and $\frac{dy}{dx}$ at the point where $x = 1$.	• Method: logarithm of each side $\ln y = \ln x^{5x^3-2} = \left(5x^3 - 2\right)\ln x$ • Logarithmic differentiation $\frac{1}{y}\frac{dy}{dx} = ...$ • Product rule $15x^2 \cdot \ln x + \left(5x^3 - 2\right)\cdot\frac{1}{x}$ • Expression for $\frac{dy}{dx}$ $\frac{dy}{dx} = y\left(15x^2 \cdot \ln x + \frac{5x^3-2}{x}\right)$ • Evaluate $y = 1, \quad \frac{dy}{dx} = 3$	**5**

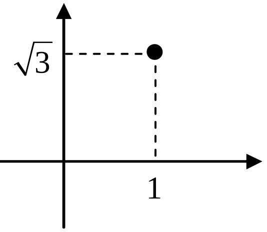
No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
11.	2.3		4	Show that *i* is a solution of $2z^3 - 3z^2 + 2z = 3$. Hence find all solutions.	• Substitute to satisfy equation $2i^3 - 3i^2 + 2i = -2i + 3 + 2i = 3$ • State first root and its conjugate $z = \pm i$ are roots • Product of factors $z^2 + 1$ is a factor • Perform division and solve $\frac{2z^3 - 3z^2 + 2z - 3}{z^2 + 1} = (2z - 3) \quad \therefore z = \pm i, \frac{3}{2}$	**4**

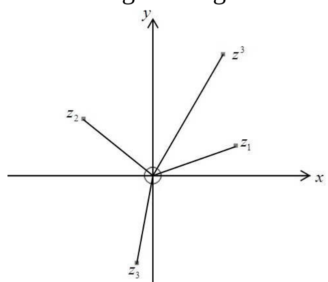
[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
12.	2.2		6	The Malthus Equation is a basic model used to predict population increases and decreases. It states $\frac{dP}{dt} = kP$ where k is a positive constant, P is the population and t is the time in years. The initial population of a town was 5 000 people. Five years later, the town's population had grown by 1 200 people. Find the value of k.	• Separate variables to integrate $\int \frac{1}{P}\, dP = \int k\ dt$ • Integrate $\ln P = kt + c$ • Simplify $P = Ae^{kt}$ • Use initial condition to find A $A = 5000$ • Know to use second condition to find k $k = \frac{1}{5}\ln\left(\frac{6200}{5000}\right)$ • State value of k $k \approx 0\cdot043$	**6**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels A/B	Marks at levels C			
13.	1.3	9		The design for a solid Christmas decoration is made by rotating the area bound by the curve $y = -4x e^{\frac{x}{2}}$ and the x-axis about the x-axis by 2π radians between the origin and $x = -3$ as shown on the diagram below.  Find the volume of plastic required to make the Christmas decoration.	• Method $V = \int_a^b \pi y^2 \ dx$ • Apply formula $V = \pi \int_{-3}^{0} \left(-4x e^{\frac{x}{2}} \right)^2 \ dx$ • Accuracy $V = 16\pi \int_{-3}^{0} x^2 e^x \ dx$ • Use integration by parts $16\pi \left[x^2 \cdot \int e^x dx - \int 2x \cdot \left(\int e^x dx \right) dx \right]$ • Accuracy $16\pi \left[x^2 \cdot e^x - 2\int x \cdot e^x \ dx \right]_{-3}^{0}$ • Use integration by parts again $16\pi \left[x^2 \cdot e^x - 2\left(x \cdot e^x - \int 1 \cdot \left(\int e^x dx \right) dx \right) \right]_{-3}^{0}$ • Accuracy $16\pi \left[x^2 \cdot e^x - 2x \cdot e^x + 2\int e^x \ dx \right]_{-3}^{0}$ • Final integration $16\pi \left[x^2 \cdot e^x - 2x \cdot e^x + 2e^x \right]_{-3}^{0}$ • Evaluate $32\pi - 272\pi e^{-3} \approx 57 \cdot 99$ cubic units	9

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
14. *(a)*	2.3		2	Plot the complex number $1+\sqrt{3}\,i$ on an Argand diagram.	• Argand diagram: correct ***R*** coordinate (or modulus, $r = 2$) • Correct ***I*** coordinate (or arg $z = \frac{\pi}{3}$) 	**2**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
14. **(Cont.)** *(b)*	2.3	7		$z^3 = 1+\sqrt{3}\,i$. Find all the roots of z^3, expressing each in the form $z = r(\cos\theta + i\sin\theta)$, clearly stating the values of r and θ. Plot z on the Argand diagram used in part (a).	• Calculate modulus and arg $r = 2,\ \theta = \frac{\pi}{3}$ • Express in polar form $z^3 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ • De Moivre's Theorem $z = 2^{\frac{1}{3}}\left(\cos\frac{1}{3}(...) + i\sin\frac{1}{3}(...)\right)$ • Know rule for n^{th} roots of unity $z = 2^{\frac{1}{3}}\left(\cos\frac{1}{3}\left(\frac{\pi}{3}+2k\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{3}+2k\pi\right)\right)$ for $k = 0, 1, 2$ • Apply rule to get first two roots $z = 2^{\frac{1}{3}}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$ and $z = 2^{\frac{1}{3}}\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right)$ • Apply rule to get final root, simplified $z = 2^{\frac{1}{3}}\left(\cos\frac{5\pi}{9} - i\sin\frac{5\pi}{9}\right)$ *or* $2^{\frac{1}{3}}\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$ • Each root plotted on Argand diagram 	7

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
15. *(a)*	2.5	2	2	Prove by induction that $\sum_{r=1}^{n} 4r(r-2) = \frac{2n}{3}(n+1)(2n-5)$ for all natural numbers $n \geq 1$.	• Show true for n = 1, assume true for n = k $n=1,\ 4n(n-2)=-4,\ \frac{2n}{3}(n+1)(2n-5)=-4$ so true for n = 1. Assume true for n = k so $\sum_{r=1}^{k} 4r(r-2) = \frac{2k}{3}(k+1)(2k-5)$ • Consider n = k + 1 with an appropriate expression $\sum_{r=1}^{k+1} 4r(r-2) = \frac{2k}{3}(k+1)(2k-5)+4(k+1)(k-1)$ • Manipulate expression $\ldots = \frac{2}{3}(k+1)(k+2)(2k-3)$ $= \frac{2}{3}(k+1)\big((k+1)+1\big)\big(2(k+1)-5\big)$ • Complete proof If true for n = k then also true for n = k + 1 and since true for n = 1, it must be true $\forall n$.	**4**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
15. (Cont.) *(b)*	2.4		2	Hence evaluate $\sum_{r=10}^{25} 4r(r-2)$.	• Evaluate for $n = 25$ $\frac{2\times 25}{3}(25+1)(2\times 25-5)=19500$ • Evaluate for $n = 9$ and state answer $\frac{2\times 9}{3}(9+1)(2\times 9-5)=780$ and so $\sum_{r=10}^{25} 4r(r-2)=19500-780=18720$	**2**

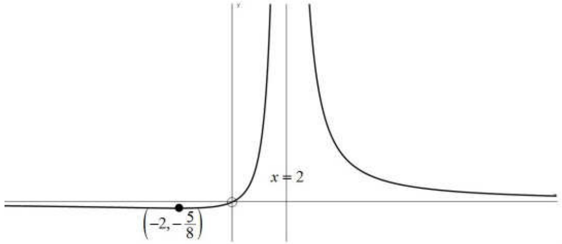
[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
16.	2.1	5	2	Given the equation $3x^2 + 6x - 6xy - 2y + 3y^2 = 5$ of a curve, obtain the *x*-coordinate for the point at which the curve has a horizontal tangent.	• Know to do implicit differentiation $6x + 6 - \left(6y + 6x\frac{dy}{dx}\right) - 2\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ • Rearrange $6x + 6 - 6y + \frac{dy}{dx}\left(-6x - 2 + 6y\right) = 0$ • Accuracy $\frac{dy}{dx} = \frac{3y - 3x - 3}{3y - 3x - 1}$ • Know to set $\frac{dy}{dx} = 0$ $3x + 3 - 3y = 0$ • Find expression for *y* in terms of *x* $y = x + 1$ • Substitute $3x^2 + 6x - 6x(x+1) - 2(x+1) + 3(x+1)^2 = 5$ • Find *x* value $x = 1$	7

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
17.	1.1		3	Express $\frac{5x}{(x-2)^2}$ in partial fractions.	• Know how to find partial fractions $\frac{A}{x-2}+\frac{B}{(x-2)^2}=\frac{5x}{(x-2)^2}$ • Find B $A(x-2)+B=5x$ $Let\ \ x=2\quad \therefore B=10$ • Find A and express as partial fractions $A(x-2)=5(x-2)\qquad \therefore A=5$ $\therefore \frac{5}{x-2}+\frac{10}{(x-2)^2}$	**3**

[Turn over

No	Analysis: Unit / Outcome	Marks at levels: A/B	Marks at levels: C	Question	Illustrations of evidence for awarding each mark	Marks
17. (Cont.) (i)	1.4		2	A curve is defined by $y = \frac{5x}{(x-2)^2}$, $(x \neq 2)$. Write down equations for its asymptotes.	• State the vertical asymptote $x = 2$ • State the non vertical asymptote $y = \dfrac{5\frac{x}{x^2}}{\frac{x^2}{x^2} - \frac{4x}{x^2} + \frac{4}{x^2}} = \dfrac{\frac{5}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}}$ and as $x \to \infty$, $y \to \frac{0}{1-0+0} = 0 \quad \therefore y = 0$	**2**
(ii)	1.2, 1.4		4	Find the stationary point and justify its nature.	• Quotient rule $\frac{dy}{dx} = \frac{5\cdot(x-2)^2 - 5x\cdot 2(x-2)}{(x-2)^4}$ • Simplify $-\frac{5x^2-20}{(x-2)^4} = -\frac{5(x+2)}{(x-2)^3}$ • Set equal to zero and solve $x^2 = 4 \quad \therefore x = \pm 2$ *but* $x \neq 2 \;\therefore x = -2$ • Find coordinates and justify nature $y = -\frac{5}{8}$, use nature table *or* second derivative to show minimum turning point at $\left(-2, -\frac{5}{8}\right)$.	**4**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
17. (Cont.) (iii)	1.4		2	Sketch the curve showing clearly the features found in (i) and (ii).	• Shape • Position 	**2**

Total 102 marks

[END OF MARKING SCHEME]

Additional Questions for unit 3

No	Analysis: Unit / Outcome	Marks at levels A/B	Marks at levels C	Question	Illustrations of evidence for awarding each mark	Marks
A1 *(a)*	Vectors 3.1		4	Find the equation of the plane $\prod$ which contains the points P(2, 1, 0), Q(–1, 0, 0) and R(1, 1, –1).	• Find two vectors in the plane $\overrightarrow{PQ} = -3\underset{\sim}{i} - \underset{\sim}{j}$ and $\overrightarrow{PR} = -\underset{\sim}{i} - \underset{\sim}{k}$ • Vector product of two vectors in the plane $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & -1 & 0 \\ -1 & 0 & -1 \end{vmatrix}$ • Accuracy • Equation of the plane $x - 3y - z = -1$ or $-x + 3y + z = 1$	**4**

No	Analysis: Unit / Outcome	Analysis: Marks at levels A/B	Analysis: Marks at levels C	Question	Illustrations of evidence for awarding each mark	Marks
A1 (Cont.) *(b)*	3.1		3	Calculate the point of intersection of the plane $\prod$ and the line L, $$\frac{x+2}{3} = y-1 = \frac{z+8}{2}.$$	• Parametric form $x = 3t-2, \quad y = t+1, \quad z = 2t-8$ • Find value for parameter $t = 3$ • Find point of intersection $(7,\ 4,\ -2)$	
	3.1	3		Find the size of the acute angle between the line L and the plane $\prod$.	• Use scalar product appropriately $\cos\theta = \dfrac{\left(\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}\right)\cdot\left(3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)}{\left(\sqrt{1^2+3^3+(-1)^2}\right)\left(\sqrt{3^2+1^2+2^2}\right)}$ • Accuracy $\dfrac{-2}{\sqrt{11}\sqrt{14}}$ • Acute angle 9.3^{o}	**6**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
B1.	3.2		6	A matrix is defined as $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ -4 & 1 & 1 \end{pmatrix}$. Show that matrix A has an inverse, A^{-1}, and use elementary row operations to find the inverse matrix.	• For a structured approach: matrix form $\begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$ • For simplifying matrix $\begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 \end{pmatrix}$ • For upper triangular form on left $\begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 10 & 0 & 2 & 3 & 1 \\ 0 & 0 & 10 & 6 & -1 & 3 \end{pmatrix}$ • For further 'reduction' on left $\begin{pmatrix} 20 & 0 & 0 & 4 & 1 & -3 \\ 0 & 10 & 0 & 2 & 3 & 1 \\ 0 & 0 & 10 & 6 & -1 & 3 \end{pmatrix}$ • Reduce left to identity matrix $\begin{pmatrix} 1 & 0 & 0 & 4/20 & 1/20 & -3/20 \\ 0 & 1 & 0 & 4/20 & 6/20 & 2/20 \\ 0 & 0 & 1 & 12/20 & -2/20 & 6/20 \end{pmatrix}$ • Correct answer $A^{-1} = \frac{1}{20}\begin{pmatrix} 4 & 1 & -3 \\ 4 & 6 & 2 \\ 12 & -2 & 6 \end{pmatrix}$	**6**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
B2.	3.2		4	The matrix *P* is such that $P^2 = 3P - 5I$ where *I* is the corresponding identity matrix. Find integers *a* and *b* such that $P^4 = aP + bI$.	• Multiply P^2 by P $P^3 = 3P^2 - 5PI$ • Simplify $4P - 15I$ • Multiply P^3 by P $P^4 = 4P^2 - 15PI$ • Simplify and state values of *a* and *b* $-3P - 20I \quad \therefore \; a = -3, \; b = -20$	**4**

[Turn over

<table>
<tr><th rowspan="3">No</th><th colspan="3">Analysis</th><th rowspan="3">Question</th><th rowspan="3">Illustrations of evidence for awarding each mark</th><th rowspan="3">Marks</th></tr>
<tr><th rowspan="2">Unit / Outcome</th><th colspan="2">Marks at levels</th></tr>
<tr><th>A/B</th><th>C</th></tr>
<tr><td>B3.</td><td>3.2</td><td>2</td><td>3</td><td>Write down the 2×2 matrix A representing a rotation of $\frac{\pi}{3}$ radians about the origin in an clockwise direction and the 2×2 matrix B representing a reflection in the y-axis.
Hence, show that the image of the point (x, y) under the transformation A followed by the transformation B is $\left(-\frac{x-py}{2}, \frac{px+y}{2}\right)$, stating the value of p.</td><td>• State rotation matrix
$\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$
• Simplify
$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
• State reflection matrix
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
• Begin calculation
$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
• Answer
$\begin{pmatrix} -\frac{x}{2} + \frac{\sqrt{3}y}{2} \\ \frac{\sqrt{3}x}{2} + \frac{y}{2} \end{pmatrix} \quad \therefore\ p = \sqrt{3}$</td><td>5</td></tr>
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No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
C1	3.3, 1.2	5		Find the Maclaurin expansion of $$f(x) = \ln(1+\sin x)\,,\ 0 < x \le \frac{\pi}{2}$$ as far as the term in x^3.	• Evaluate $f(0)$, differentiate and evaluate $f'(0)$ $f(0) = 0,\ f'(x) = \frac{\cos x}{1+\sin x}\ \therefore\ f'(0) = 1$ • Differentiate and evaluate $f''(0)$ $f''(x) = \frac{-\sin x \cdot (1+\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2}$ $= \frac{-\sin x - 1}{(1+\sin x)^2}\quad \therefore\ f''(0) = -1$ • Differentiate and evaluate $f'''(0)$ $f'''(x) =$ $\frac{-\cos x(1+\sin x)^2 + (\sin x + 1)(1+\sin x)\cos x}{(1+\sin x)^4}$ $\therefore\ f'''(0) = 1$ • Know Maclaurin's Series $f(x) =$ $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots$ • Substitute correctly and simplify $x - \frac{x^2}{2} + \frac{x^3}{6}$	**5**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
C2.	3.3		3	The equation $y = x^3 + x - 5$ has root between $x = 1$ and $x = 2$. Using the iterative recurrence formula $x_{n+1} = \sqrt[3]{5 - x}$ and $x_0 = 1$ find the value of the root correct to 3 decimal places.	• Use iterative process $x_1 = 1\cdot5874,\ x_2 = 1\cdot5055,\ x_3 = 1\cdot5175$ • Show that convergence occurs • $x_4 = 1\cdot5158,\ x_5 = 1\cdot5160,\ x_6 = 1\cdot5160$ • Know how to test for convergence $g'(x) = -\frac{1}{3}(5 - x)^{-\frac{2}{3}}$, $g'(1\cdot516) = -0\cdot145$ or continuing iteration for three more terms or by using a diagram • Accuracy and result $\lvert g'(1\cdot516)\rvert < 1$ so the root is $1\cdot516$ to 3 decimal places	**5**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
D1.	3.4	10		Solve the differential equation $$2\frac{d^2y}{dx^2}+\frac{dy}{dx}-3y=2\sin x.$$ given that there is a stationary point at (0, 1).	• Create auxiliary equation $2m^2+m-3=0$ • Solve auxiliary equation $m=-\frac{3}{2},1$ • State complementary function $y=Ae^{-\frac{3}{2}x}+Be^x$ $y=C\sin x+D\cos x,\ \frac{dy}{dx}=C\cos x-D\sin x$ $\frac{d^2y}{dx^2}=-C\sin x-D\cos x$ • Create particular integral and differentiate, twice $(-5C-D)\sin x+(C-5D)\cos x=2\sin x$ • Substitute into differential equation $C=-\frac{5}{13},\ D=-\frac{1}{13}$	

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
D1. (Cont.					• Obtain values for constants $y = Ae^{-\frac{3}{2}x} + Be^{x} - \frac{5}{13}\sin x - \frac{1}{13}\cos x$ • State general solution $A + B = \frac{14}{13}$ • Substitute in first set of values $-\frac{3}{2}A + B = \frac{5}{13}$ • Substitute in second set of values • Calculate values of constants $A = \frac{18}{65},\ B = \frac{4}{5}\ \ and\ \ so$ $y = \frac{18}{65}e^{-\frac{3x}{2}} + \frac{4}{5}e^{x} - \frac{5}{13}\sin x - \frac{1}{13}\cos x$	**10**

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
E1.	3.5		4	Use direct proof to show that $\sum_{r=1}^{n} 4r(r-2) = \frac{2n}{3}(n+1)(2n-5)$.	• Separate into to a sum of parts $\sum_{r=1}^{n} 4r(r-2) = 4\sum_{r=1}^{n} r^2 - 8\sum_{r=1}^{n} r$ • Know formula for $\sum_{r=1}^{n} r^2$ $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ • Know formula for $\sum_{r=1}^{n} r$ $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ • Simplify $\frac{4n}{3}(n+1)(2n+1) + 4n(n+1)$	**4**

[Turn over

No	Analysis			Question	Illustrations of evidence for awarding each mark	Marks
	Unit / Outcome	Marks at levels				
		A/B	C			
E2.	3.5	4		Use the Euclidean Algorithm to find integers x and y such that $159x + 127y = 1$.	• Use the Euclidean Algorithm to show GCD is 1 $159 = 127 \times 1 + 32$ $127 = 32 \times 3 + 31$ $32 = 31 \times 1 + 1$ • Begin back substitution $1 = 32 - 31 \times 1$ $= 32 - (127 - 32 \times 3) \times 1$ • Complete back substitution $1 = 32 \times 4 - 127 \times 1$ $= (159 - 127 \times 1) \times 4 - 127 \times 1$ $= 159 \times 4 - 127 \times 4 - 127 \times 1$ • Answer $159 \times 4 - 127 \times 5 \quad \therefore\ x = 4,\ \ y = -5$ Accept answers of the form $x = 127p + 4$, $y = -159p - 5$ for any integer p	**4**

Total 50 marks

[END OF MARKING SCHEME FOR ADDITIONAL QUESTIONS]